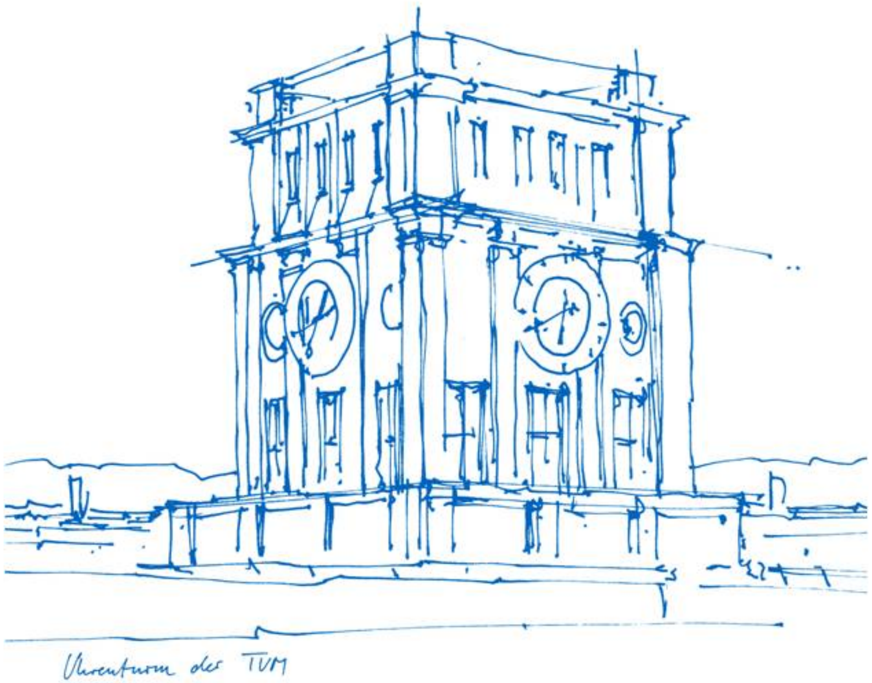
AM
Lehrstuhl für
Angewandte Mechanik

EXPERTISE

Domain Decomposition from mechanical perspective

Prof. Daniel J. Rixen

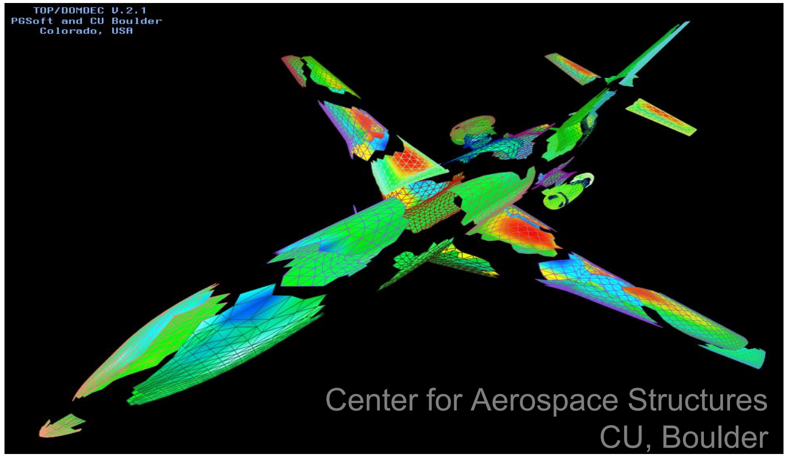
Kick-off training, January 2018
Garching



MARIE CURIE ACTIONS

European Training Network
grant agreement No 721865

Introduction



When splitting the problem in parts and asking different cpu's (or threads) to take care of subproblems, will the problem be solved faster ?

FETI, Primal Schur (Balancing) method

around 1990 …….. basic methods, mesh decomposer technology

1995-2000 …….. improvements
- preconditioners, coarse grids
- application to Helmholtz, dynamics, non-linear ...

2000-today …….. heterogeneous, dynamic, non-linear problems

Here the concepts are outlined using some mechanical interpretation.

Content

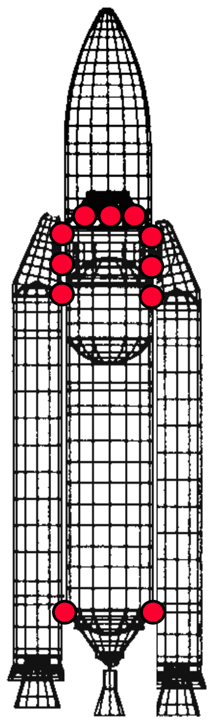
Part 1

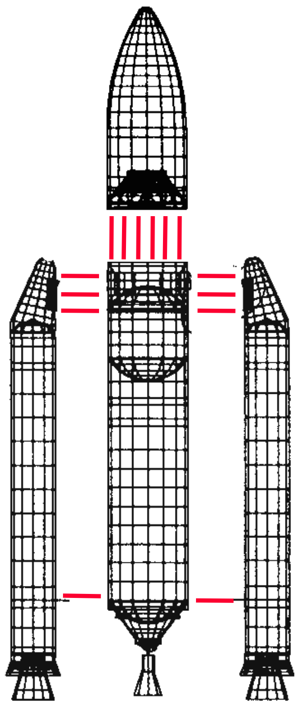
1. Domain decomposition: Primal and Dual approaches
2. The FETI algorithm and preconditioners

Part 2

3. cyclic symmetry: harmonic decomposition
4. Quasi-cyclic problems and approaches
5. Project WP3 – 1 (ESR9)

Domain Decomposition: Primal / Dual Assembly



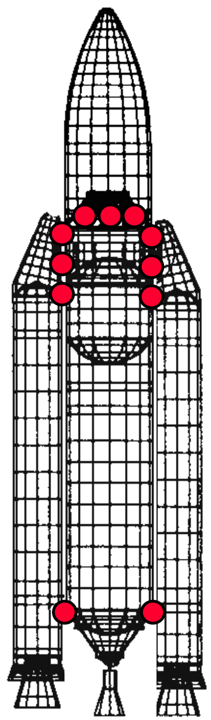


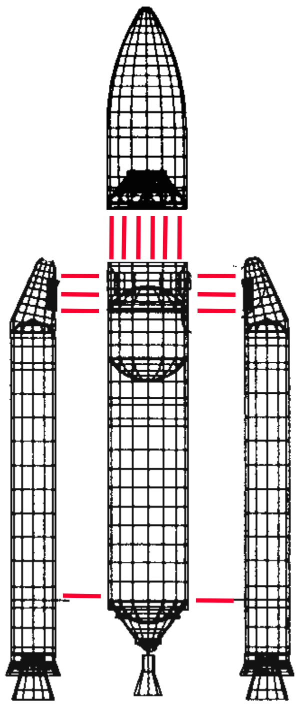
$$\begin{bmatrix} K^{(1)} & & 0 \\ & \ddots & \\ 0 & & K^{(N_s)} \end{bmatrix} \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(N_s)} \end{bmatrix} = \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(N_s)} \end{bmatrix}$$

signed Boolean matrices

$$\begin{bmatrix} K^{(1)} & & 0 & B^{(1)^T} \\ & \ddots & & \vdots \\ 0 & & K^{(N_s)} & B^{(N_s)^T} \\ B^{(1)} & \dots & B^{(N_s)} & 0 \end{bmatrix} \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(N_s)} \\ \lambda \end{bmatrix} = \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(N_s)} \\ 0 \end{bmatrix}$$

Domain Decomposition: Primal / Dual Assembly



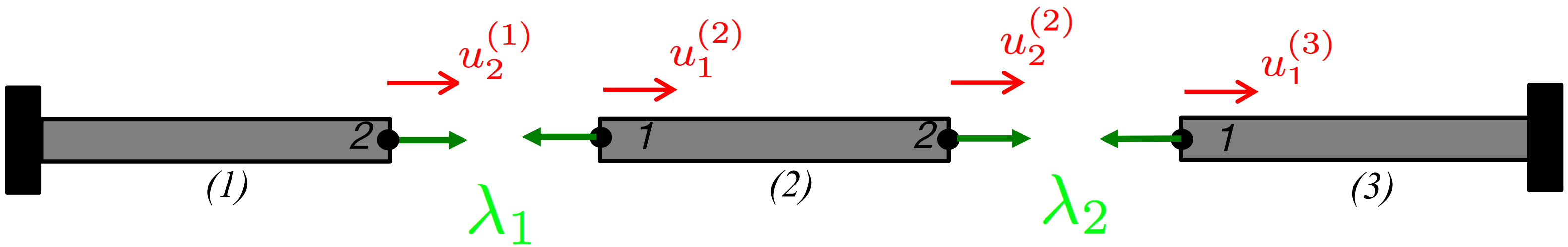


$$\begin{bmatrix} \boldsymbol{K}^{(1)} & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{K}^{(N_s)} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^{(1)} \\ \vdots \\ \boldsymbol{u}^{(N_s)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}^{(1)} \\ \vdots \\ \boldsymbol{f}^{(N_s)} \end{bmatrix}$$

Block-diagonal notation

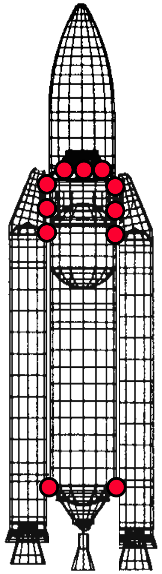
$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{B}^T \\ \boldsymbol{B} & \mathbf{o} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \mathbf{o} \end{bmatrix}$$

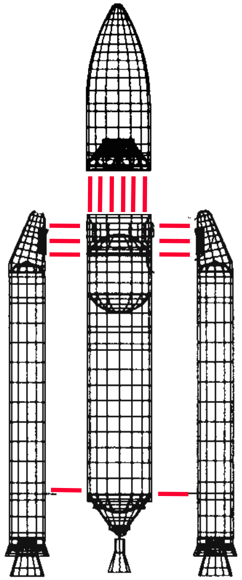
- Example of bars.



$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{B} \\ \boldsymbol{B}^T & \mathbf{o} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} k^{(1)} & 0 & 0 & 0 \\ 0 & k^{(2)} & -k^{(2)} & 0 \\ 0 & -k^{(2)} & k^{(2)} & 0 \\ 0 & 0 & 0 & k^{(3)} \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} & \mathbf{o} \end{bmatrix} \begin{bmatrix} u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ \lambda_1 \\ \lambda_2 \end{bmatrix}$$

Iterative solution of interface problem





Iterate on interface dofs $\mathbf{u}_b$

solve for internal dofs $\mathbf{u}_i$

end *(when interface in equilibrium)*

Iterate on interface forces $\boldsymbol{\lambda}$

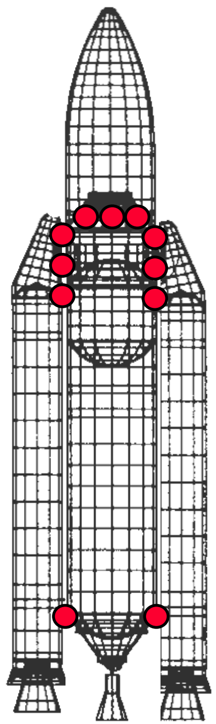
solve for domain dofs $\mathbf{u}$

end *(when interface compatible)*

Primal Schur and BDD
[Letallec *et al.*, 91]

FETI
(Finite Element Tearing and Interconnecting)
[Farhat- Roux, 91]

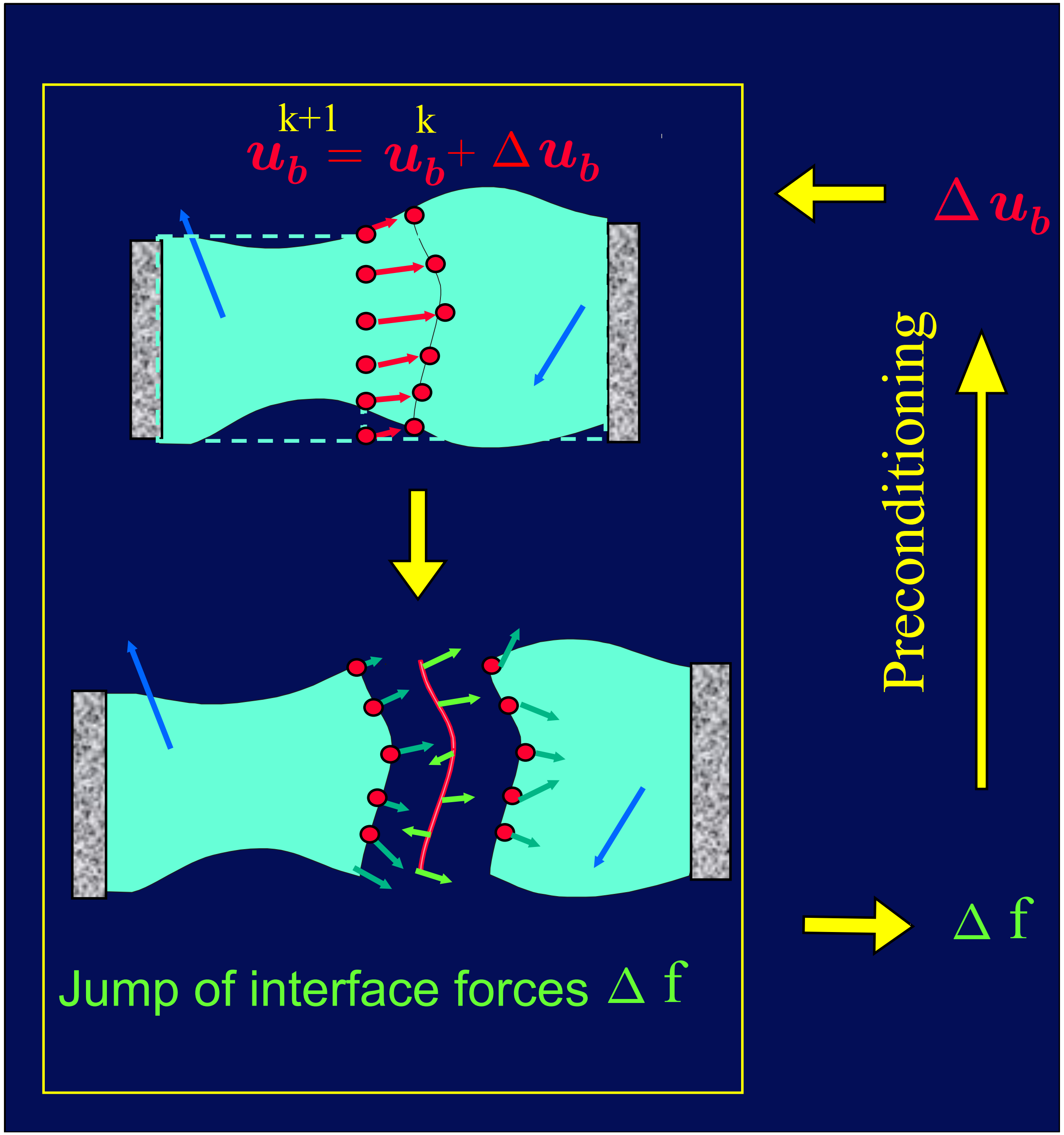
Primal Schur: iteration



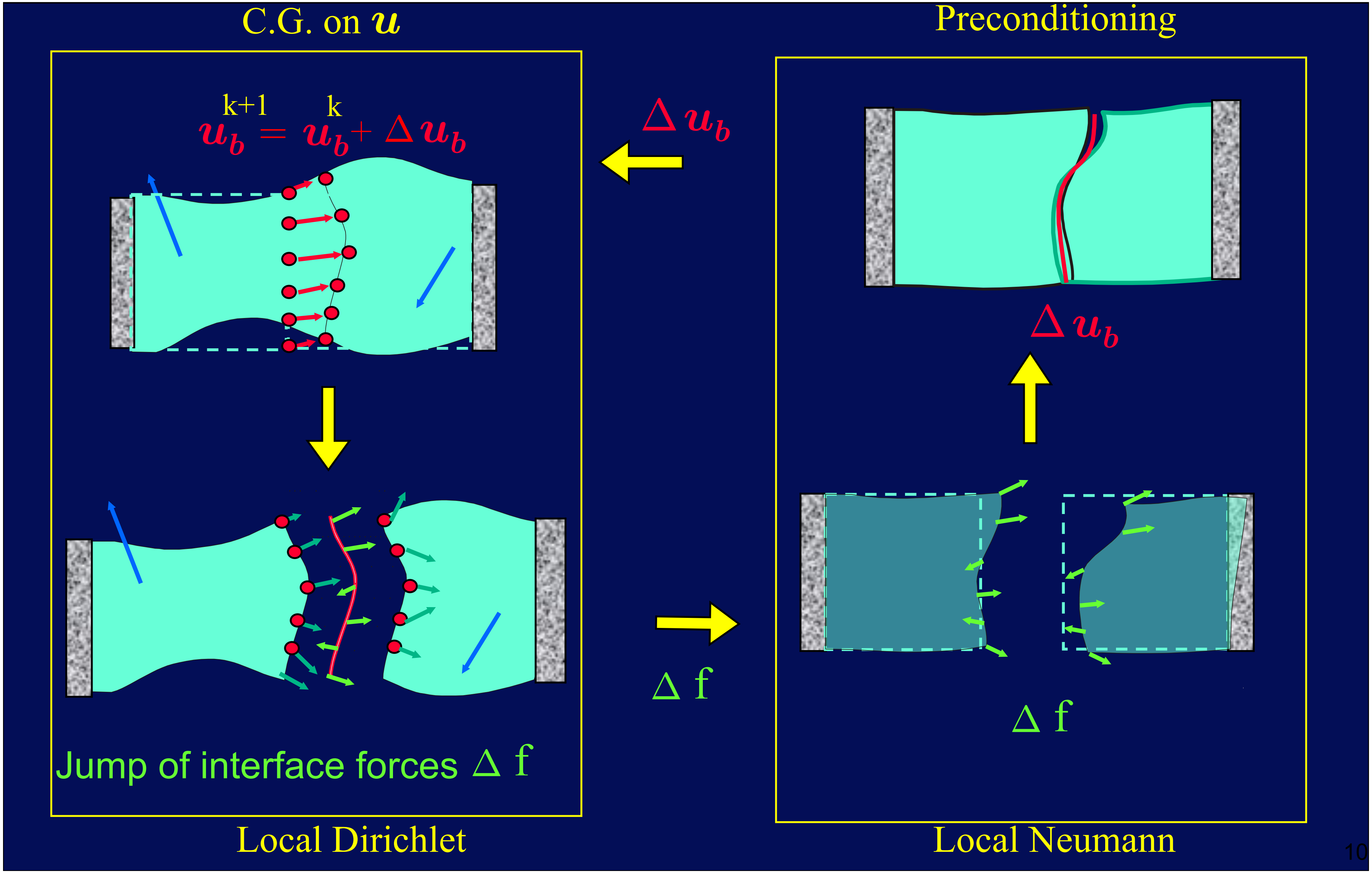
Iterate on interface dof $\boldsymbol{u}_b$

| Solve for internal dof $\boldsymbol{u}_i$

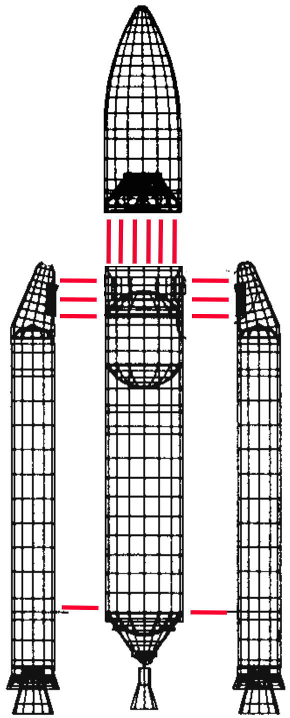
end *(when interface in equilibrium)*



Primal Schur: iteration



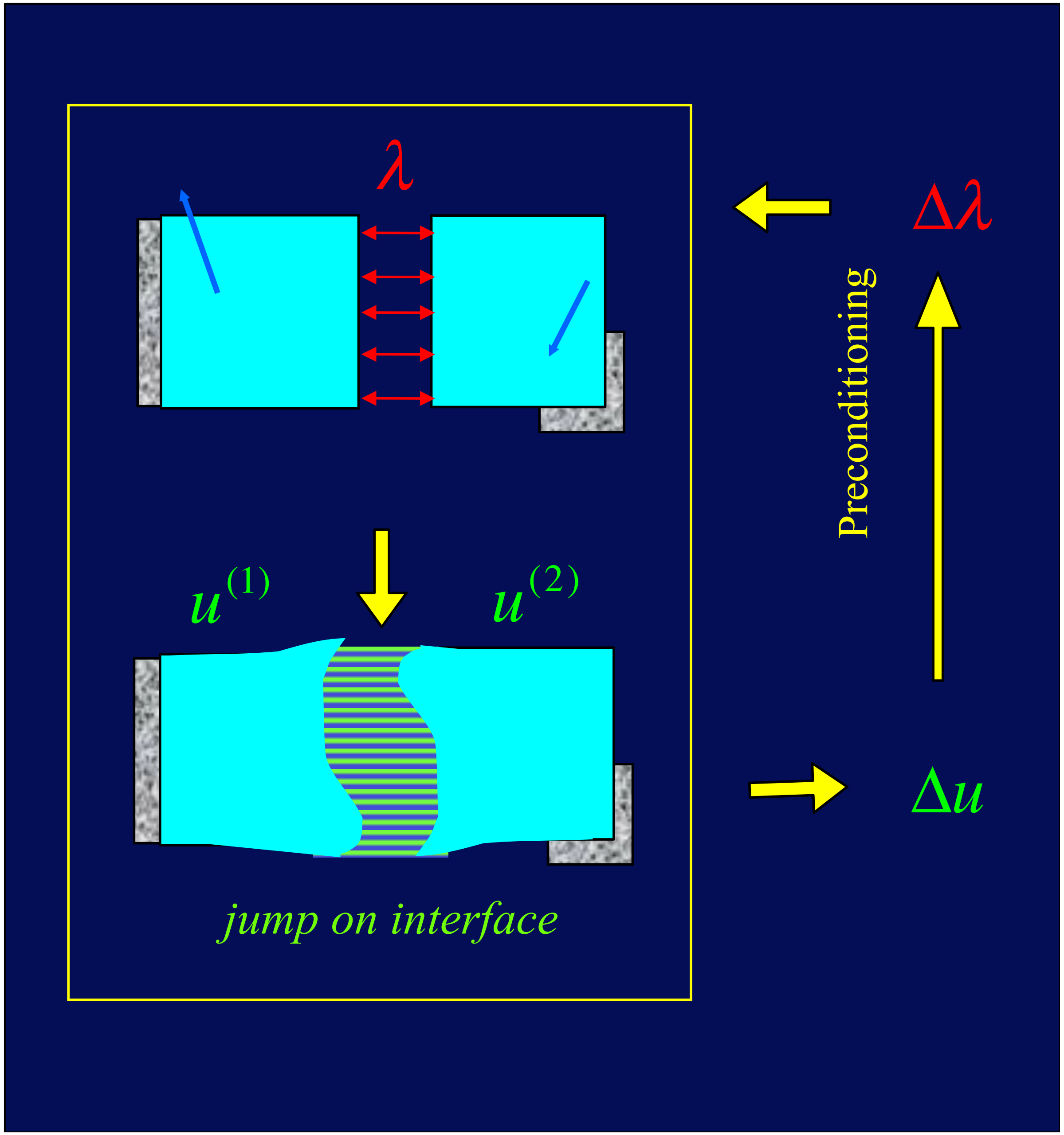
Dual Schur (FETI): iteration



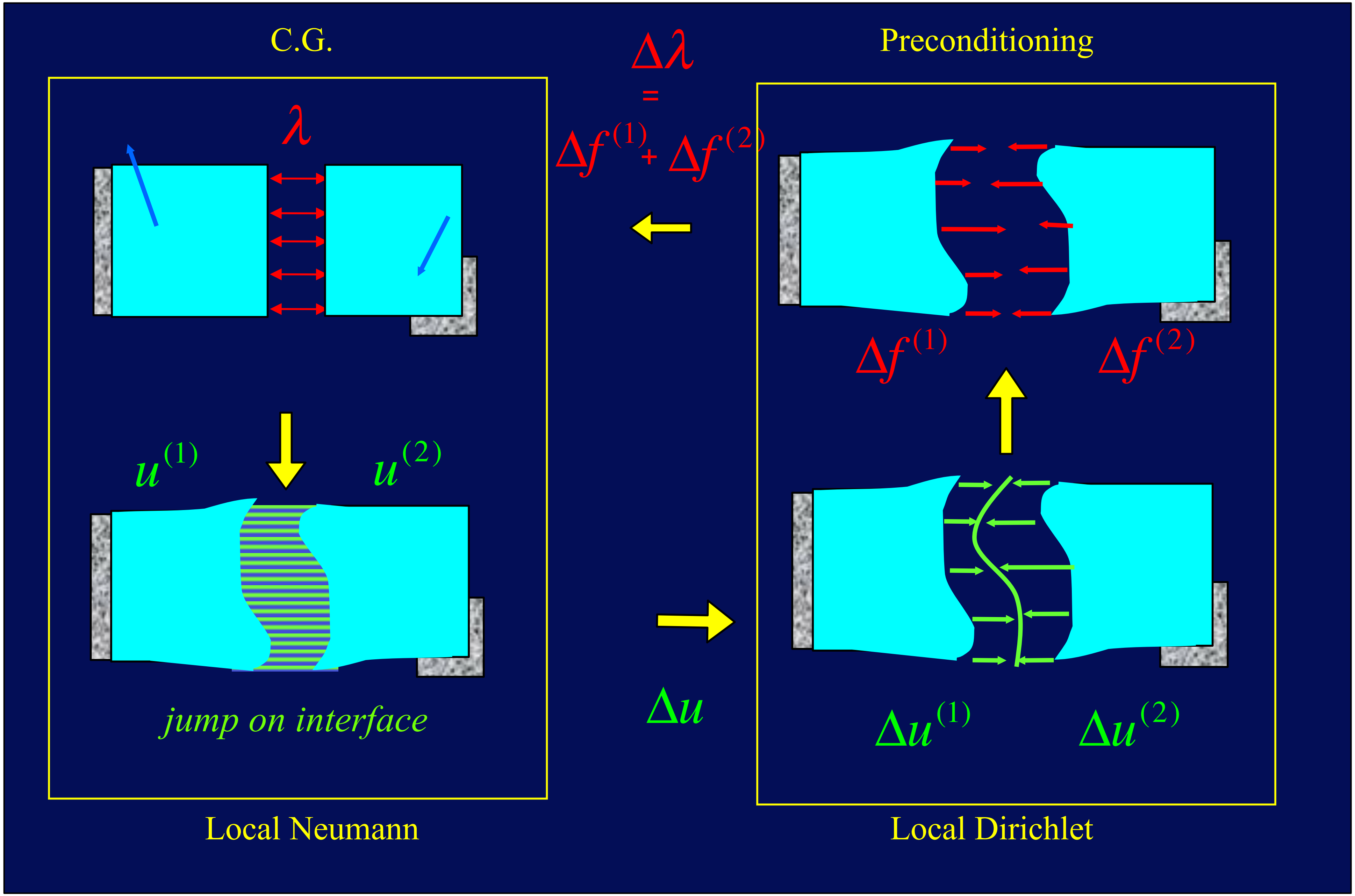
Iterate on interface forces $\boldsymbol{\lambda}$

solve for domain dofs $\mathbf{u}$

end *(when interface compatible)*



Dual Schur (FETI): iteration



The primal and dual Schur complement approaches are very similar.

Most of the "tricks" used in the one method can be applied for the other.
So which method to use is nearly a matter of religion …

There exists some subtle differences such as

- treatment of cross-points (interface nodes on more then 2 domains)
- determination of an initial estimate

[Klawonn 02, Gosselet *et al.* 03, Gosselet-Rey 06]

Here we outline only the dual approach (FETI)
but one finds many publications on similar developments for the primal approach.

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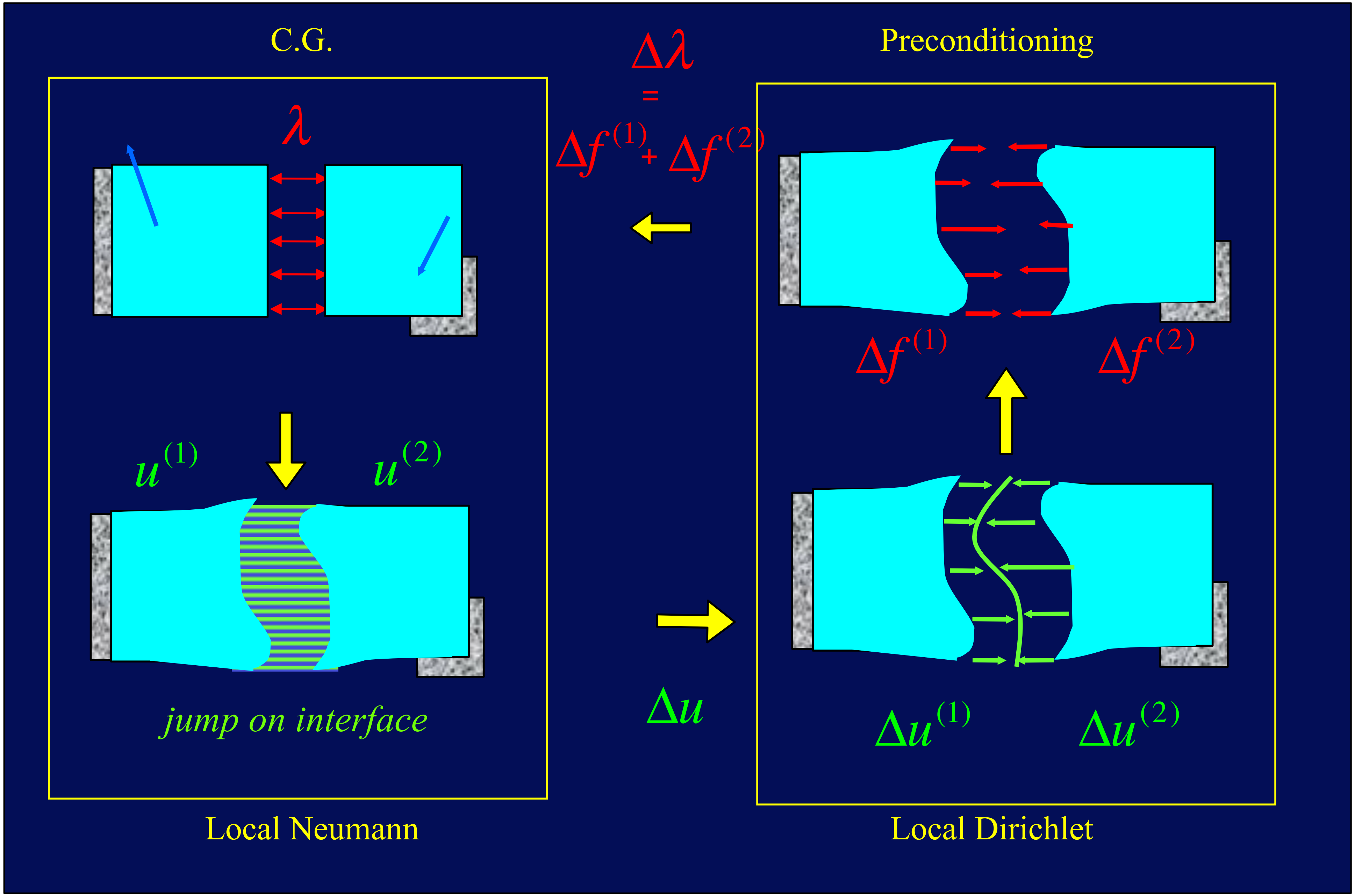
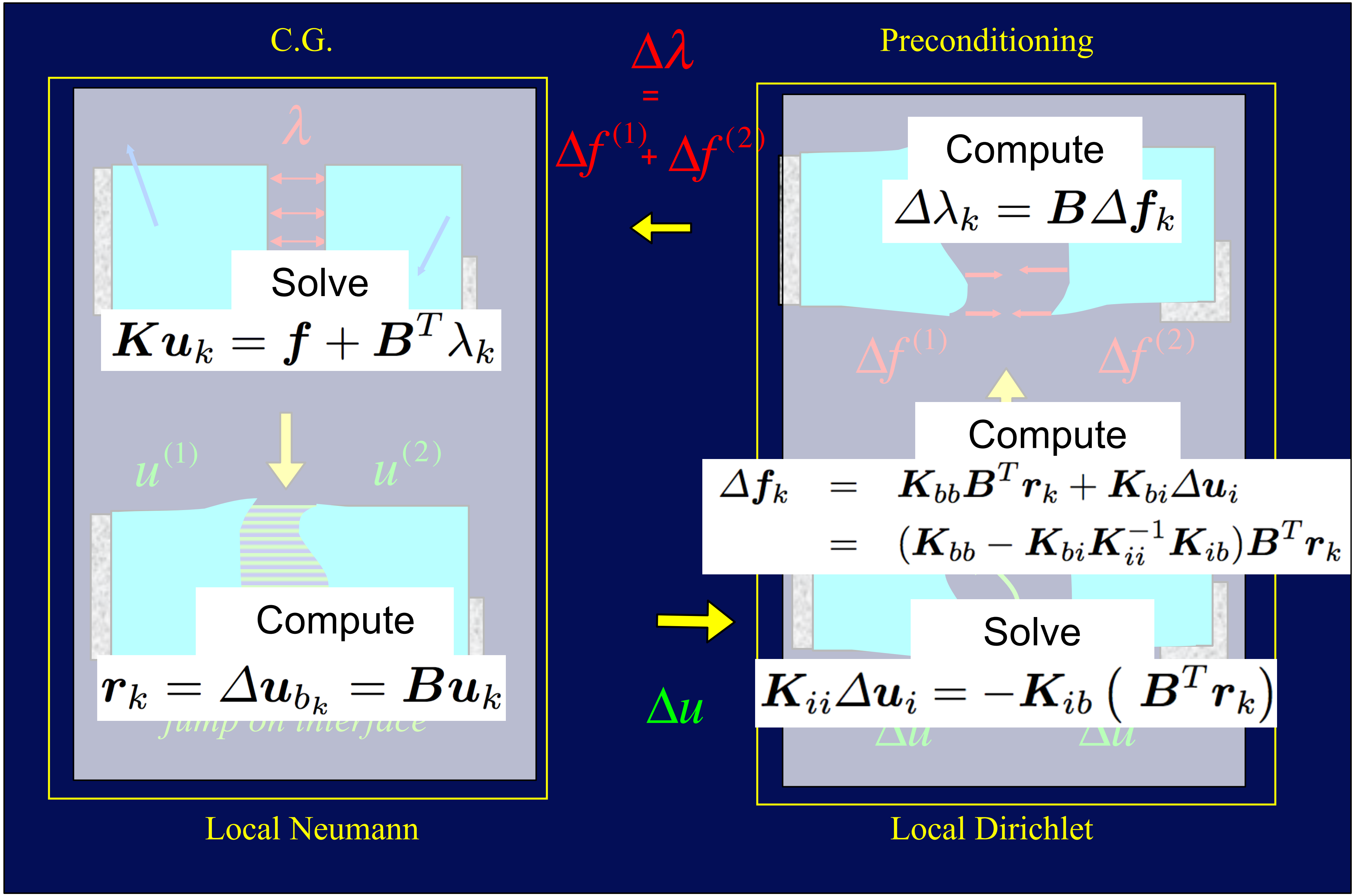
Dual Schur (FETI): iteration



Dual Schur (FETI): iteration



Dual Schur (FETI): iteration

C.G.

$\Delta\lambda = \Delta f^{(1)} + \Delta f^{(2)}$

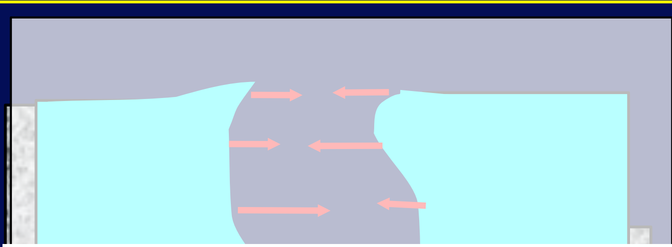
Preconditioning

Iterate on

$$\left(\boldsymbol{B}\boldsymbol{K}^{-1}\boldsymbol{B}^T\right)\lambda = \boldsymbol{B}\boldsymbol{K}^{-1}\boldsymbol{f}$$

or equivalently, considering $\boldsymbol{S}$
Schur complement on interface

$$\left(\boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{B}^T\right)\lambda = \boldsymbol{B}\boldsymbol{K}^{-1}\boldsymbol{f}$$



Precondition with

$$\Delta\lambda_k = \left(\boldsymbol{B}\boldsymbol{S}\boldsymbol{B}^T\right)\boldsymbol{r}_k$$

F_I interface flexibility

$$\underbrace{\left(\boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{B}^T\right)}_{F_I}\lambda \simeq \left(\boldsymbol{B}\boldsymbol{S}\boldsymbol{B}^T\right)^{-1}\lambda$$

„ approximate the sum of the inverse by the inverse of the sum „

jump on interface

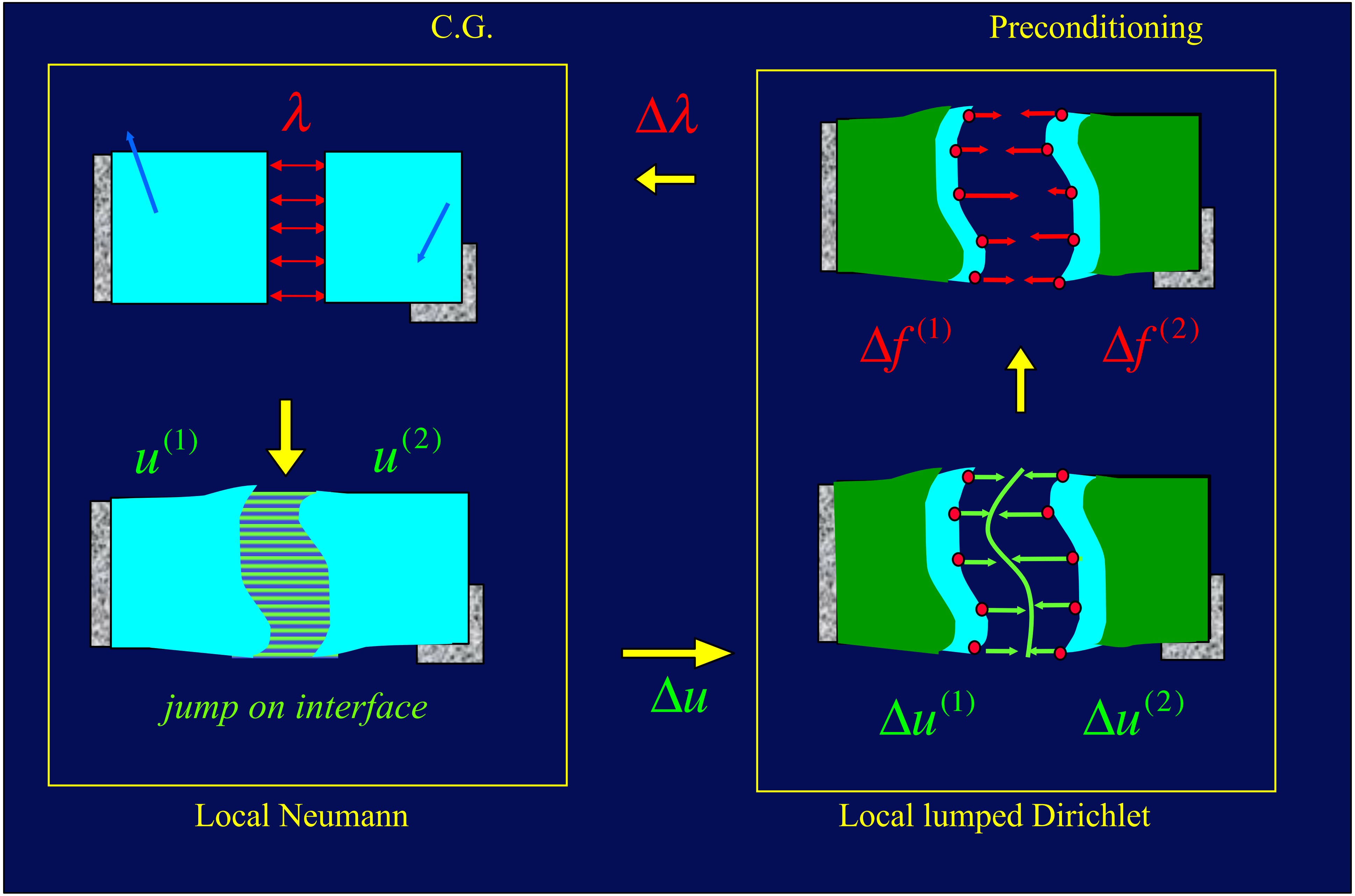
Δu

$\Delta u^{(1)}$ $\Delta u^{(2)}$

Local Neumann

Local Dirichlet

Dual Schur (FETI): lumped preconditioner



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Dual Schur (FETI): lumped preconditioner

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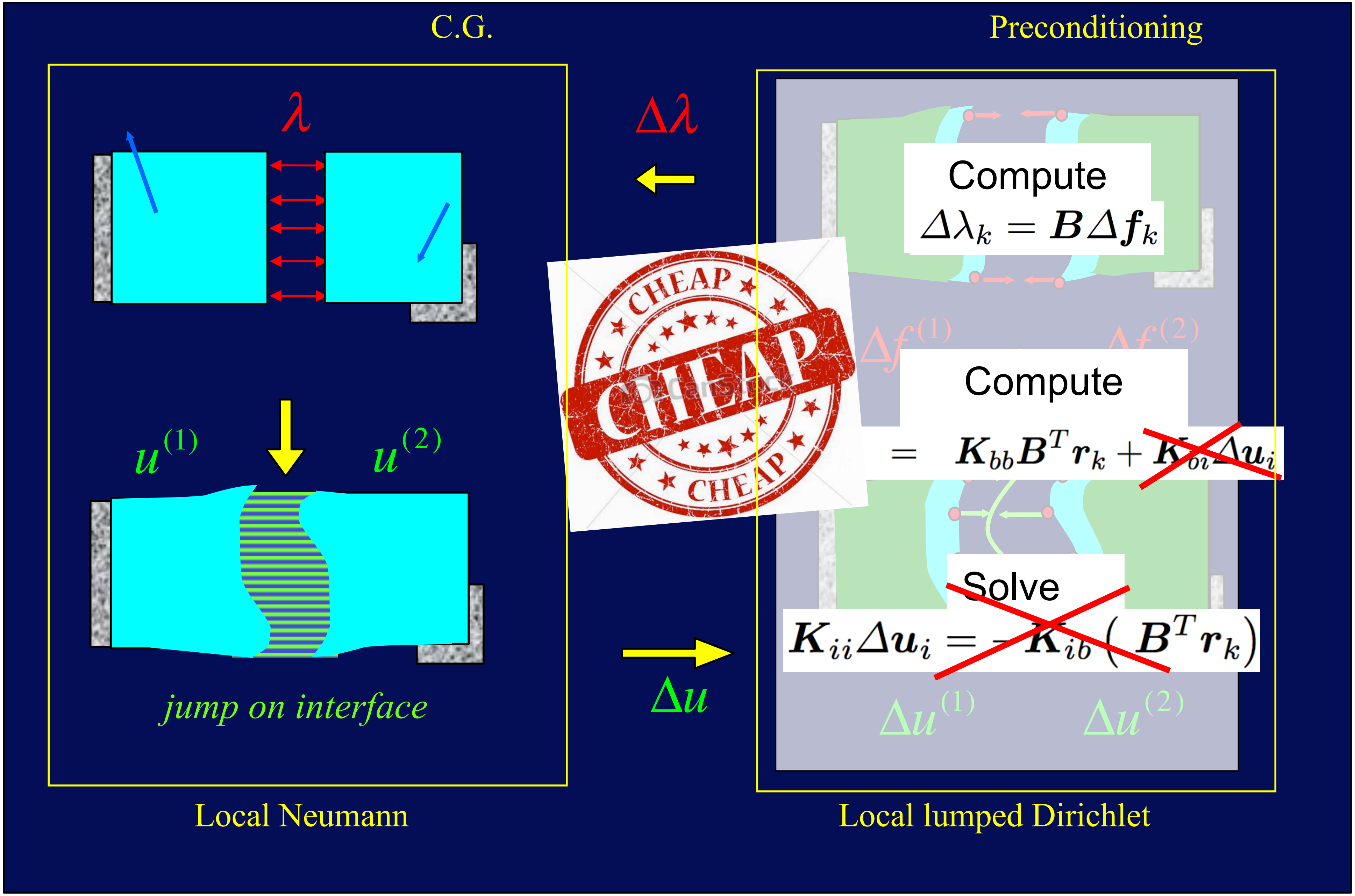
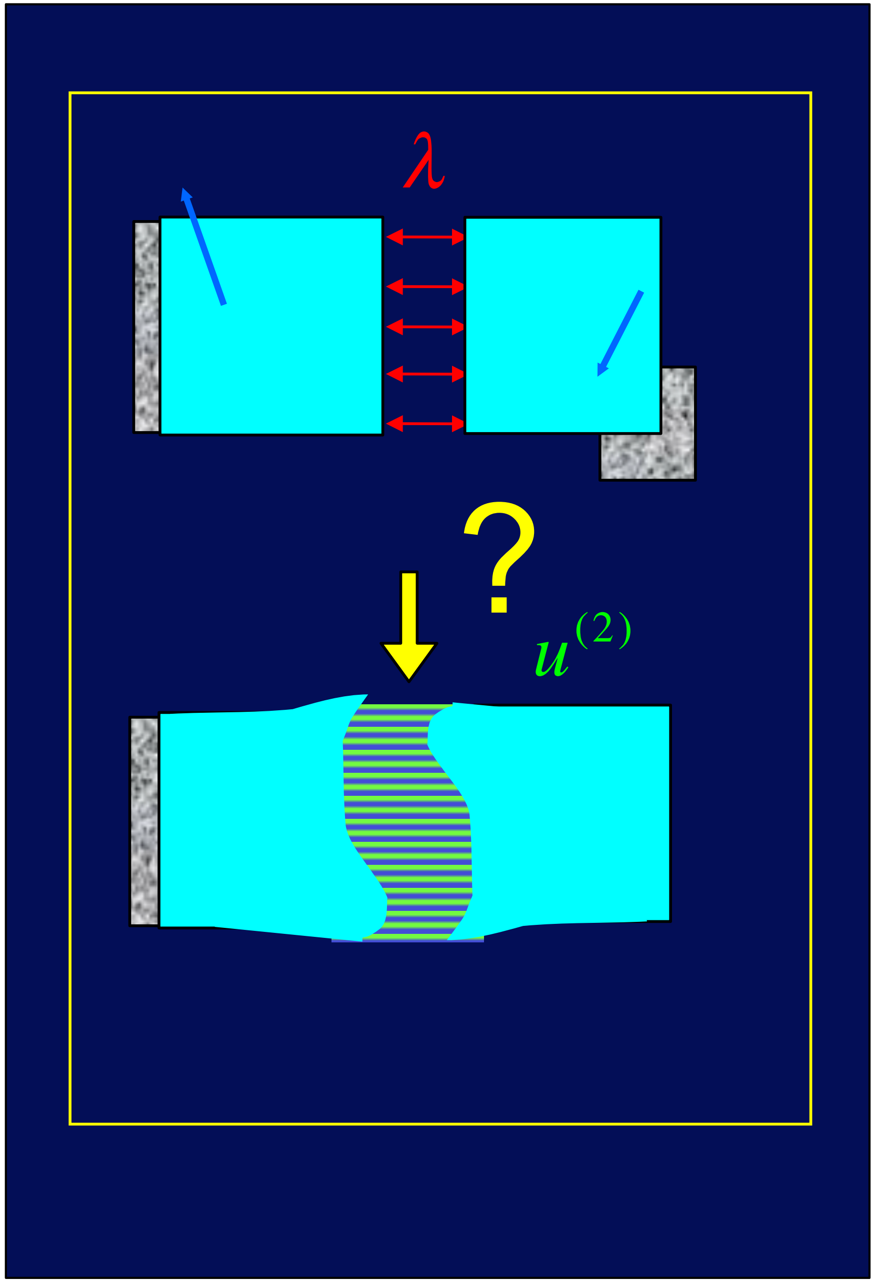


The basic FETI and its natural coarse grid



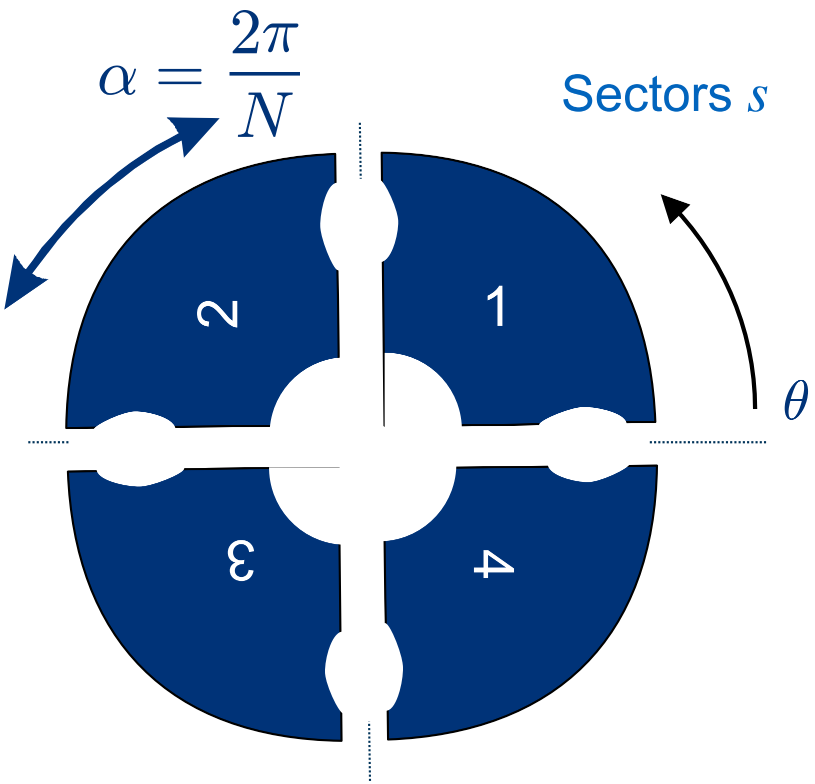
$$\begin{bmatrix} \mathbf{K} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}$$

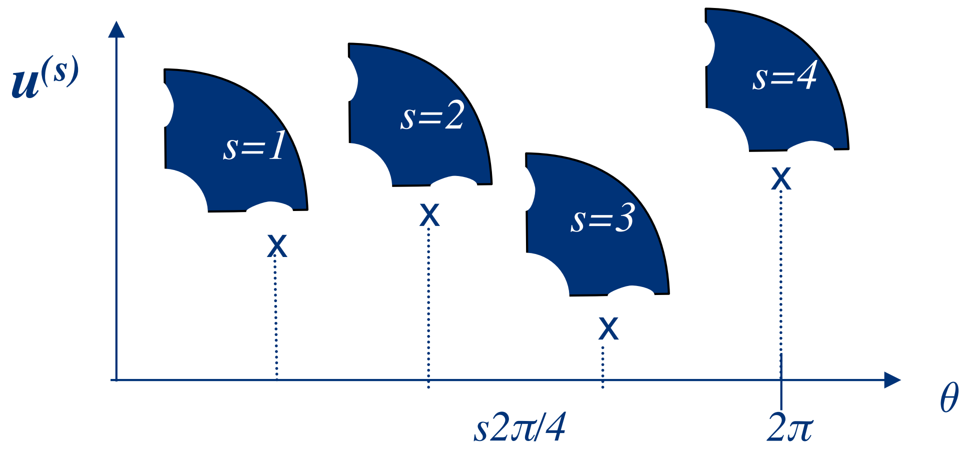
Not enough constraints
Badly defined local problems
Singular K

Requires to treat the rigid body motion of the domains separatly and leads to a so-called ***coarse grid***.
(not discussed here)

Content

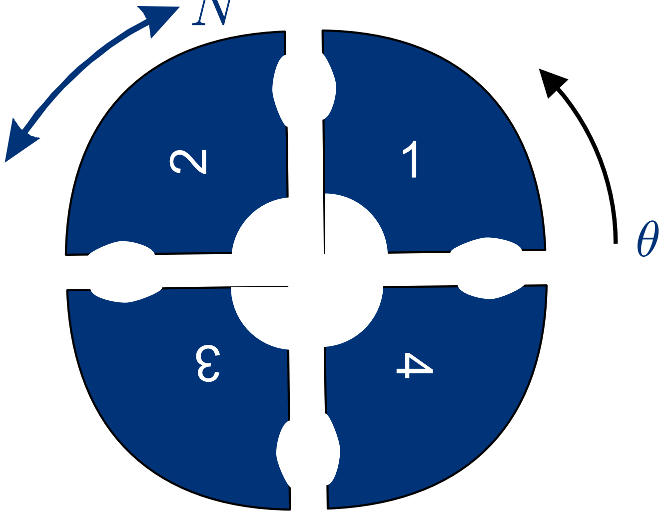
Cyclic systems: harmonic decomposition

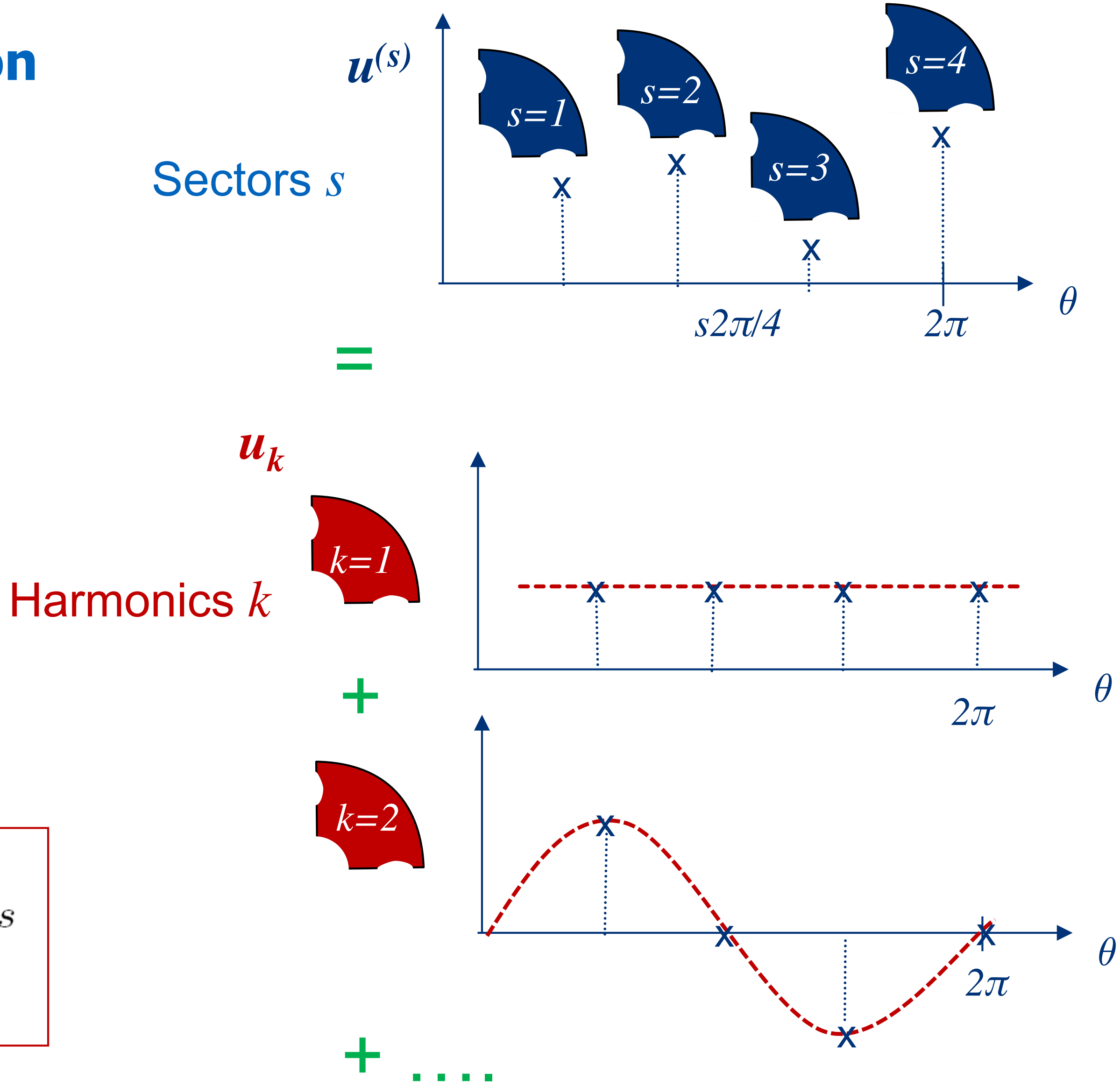




Cyclic systems: harmonic decomposition

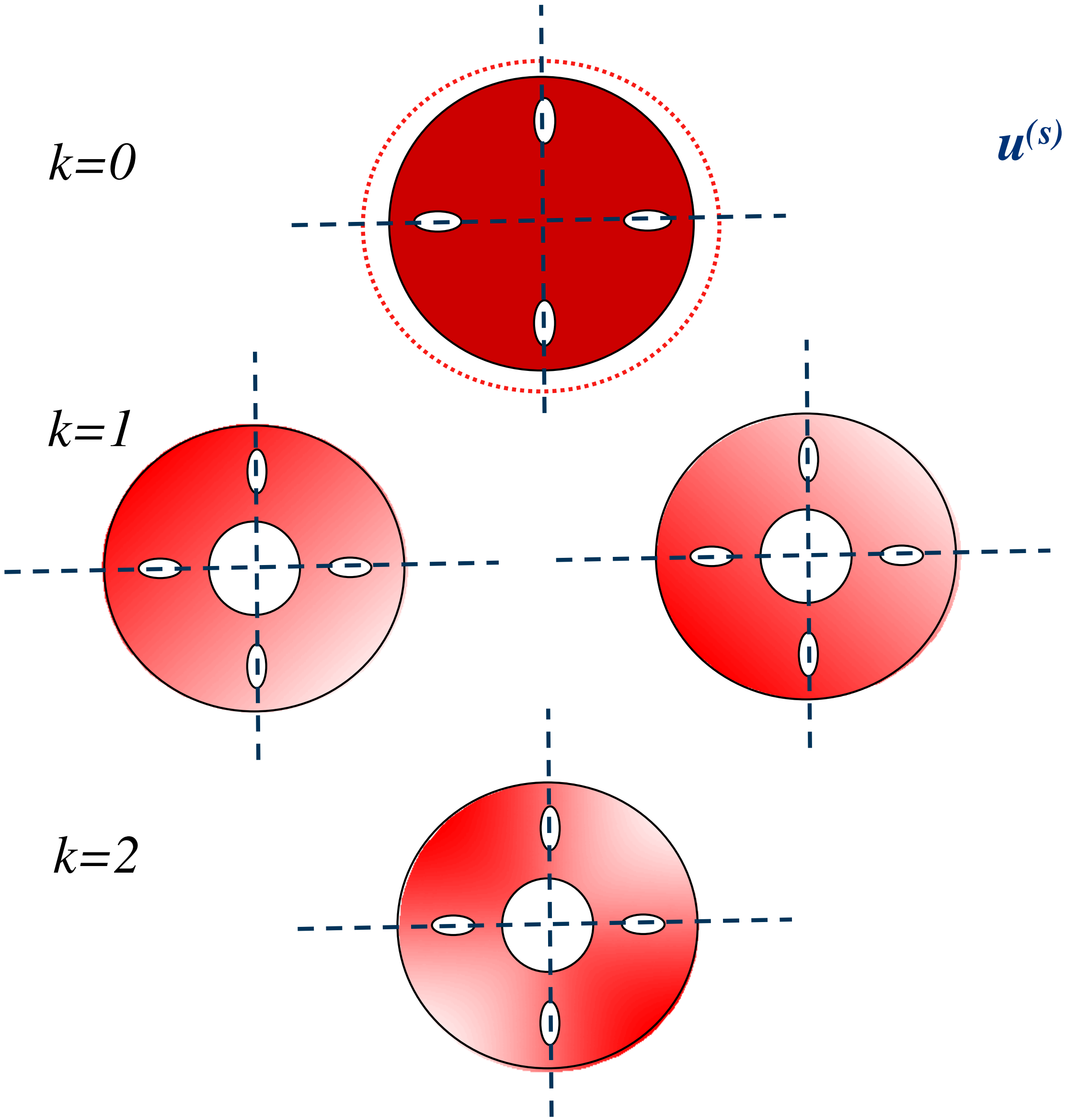
$$\alpha = \frac{2\pi}{N}$$

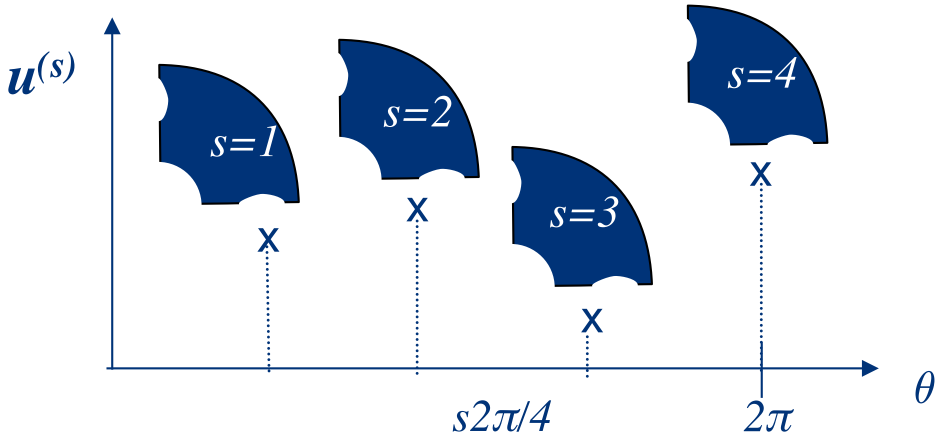




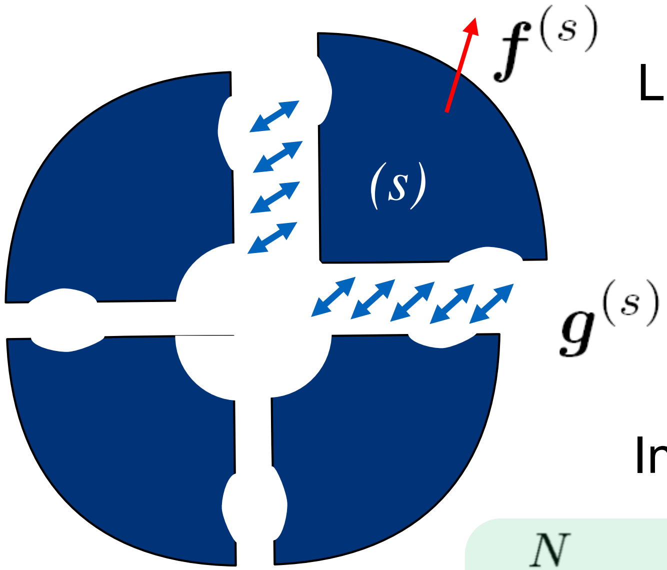
$$\boldsymbol{u}^{(s)} = \sum_{k=0}^{N-1} \boldsymbol{u}_k e^{-i\frac{k2\pi}{N}s}$$

Cyclic systems: harmonic decomposition





$$\boldsymbol{u}^{(s)} = \sum_{k=0}^{N-1} \boldsymbol{u}_k e^{-i\frac{k2\pi}{N}s}$$



Local equilibrium

$$\boldsymbol{M}^{\cancel{(s)}}\ddot{\boldsymbol{u}}^{(s)} + \boldsymbol{K}^{\cancel{(s)}}\boldsymbol{u}^{(s)} = \boldsymbol{f}^{(s)} + \boldsymbol{g}^{(s)}$$

same local matrices for each sector
but excitation does not need to be cyclic

Introducing the Fourier decomposition of sectors

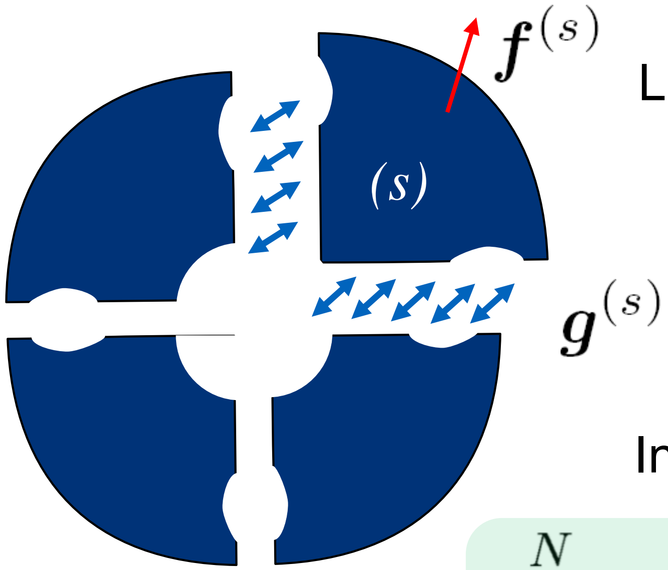
$$\sum_{s=1}^{N} e^{i\frac{r2\pi}{N}s}\left[\boldsymbol{M}\ddot{\boldsymbol{u}}^{(s)} + \boldsymbol{K}\boldsymbol{u}^{(s)} = \boldsymbol{f}^{(s)} + \boldsymbol{g}^{(s)}\right]$$

$$\sum_{k=0}^{N-1}\ddot{\boldsymbol{u}}_k e^{-i\frac{k2\pi}{N}s} \qquad \sum_{k=0}^{N-1}\boldsymbol{u}_k e^{-i\frac{k2\pi}{N}s}$$

$$\sum_{s=1}^{N} e^{-i\frac{k2\pi}{N}s}e^{i\frac{r2\pi}{N}s} = 0 \text{ if } k \neq r$$

$$= N \text{ if } k = r$$

$$\Rightarrow \boldsymbol{M}\ddot{\boldsymbol{u}}_k + \boldsymbol{K}\boldsymbol{u}_k = \boldsymbol{f}_k + \boldsymbol{g}_k$$



Local equilibrium

$$\boldsymbol{M}^{(s)}\ddot{\boldsymbol{u}}^{(s)} + \boldsymbol{K}^{(s)}\boldsymbol{u}^{(s)} = \boldsymbol{f}^{(s)} + \boldsymbol{g}^{(s)}$$

same local matrices for each sector
but excitation does not need to be cyclic

Introducing the Fourier decomposition of sectors

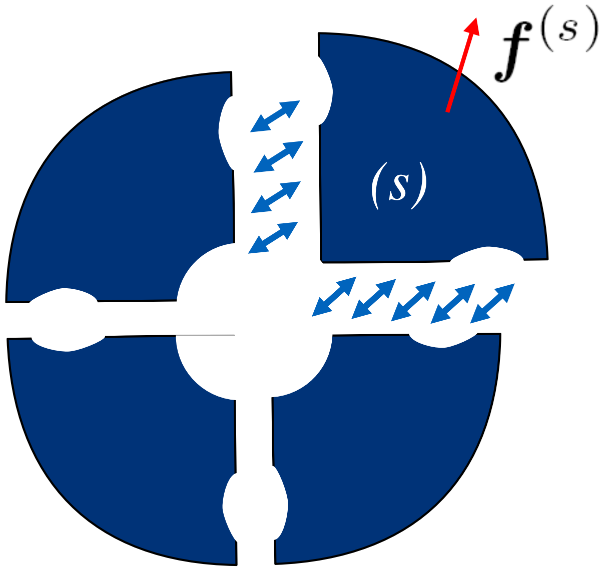
$$\sum_{s=1}^{N} e^{i\frac{r2\pi}{N}s}\left[\boldsymbol{M}\ddot{\boldsymbol{u}}^{(s)} + \boldsymbol{K}\boldsymbol{u}^{(s)} = \boldsymbol{f}^{(s)} + \boldsymbol{g}^{(s)}\right]$$

$$\ddot{\boldsymbol{u}}^{(s)} = \sum_{k=0}^{N-1} \ddot{\boldsymbol{u}}_k e^{-i\frac{k2\pi}{N}s} \qquad \boldsymbol{u}^{(s)} = \sum_{k=0}^{N-1} \boldsymbol{u}_k e^{-i\frac{k2\pi}{N}s}$$

From the orhtogonality of harmonic functions:

$$\begin{aligned}\sum_{s=1}^{N} e^{-i\frac{k2\pi}{N}s} e^{i\frac{r2\pi}{N}s} &= 0 \text{ if } k \neq r \\ &= N \text{ if } k = r\end{aligned}$$

$$\Rightarrow \quad \boldsymbol{M}\ddot{\boldsymbol{u}}_k + \boldsymbol{K}\boldsymbol{u}_k = \boldsymbol{f}_k + \boldsymbol{g}_k$$



Uncoupled analysis of 1 sector for each harmonic

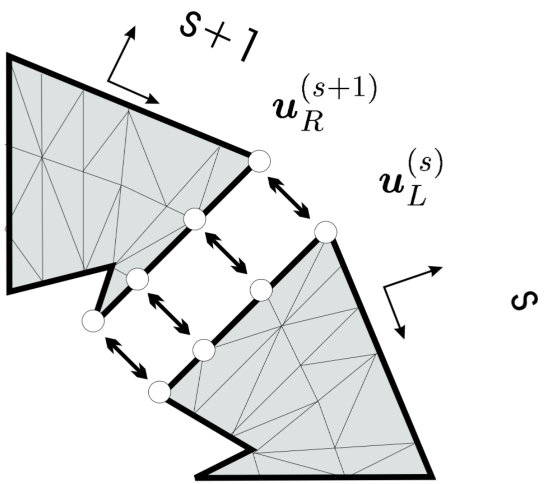
$$\boldsymbol{M}\ddot{\boldsymbol{u}}_k + \boldsymbol{K}\boldsymbol{u}_k = \boldsymbol{f}_k + \boldsymbol{g}_k$$

where

$$\boldsymbol{f}_k = \frac{1}{N}\sum_{s=1}^{N} \boldsymbol{f}^{(s)} e^{i\frac{k2\pi}{N}s}$$

$$\boldsymbol{g}_k = \frac{1}{N}\sum_{s=1}^{N} \boldsymbol{g}^{(s)} e^{i\frac{k2\pi}{N}s}$$

are the Fourier components of the forces

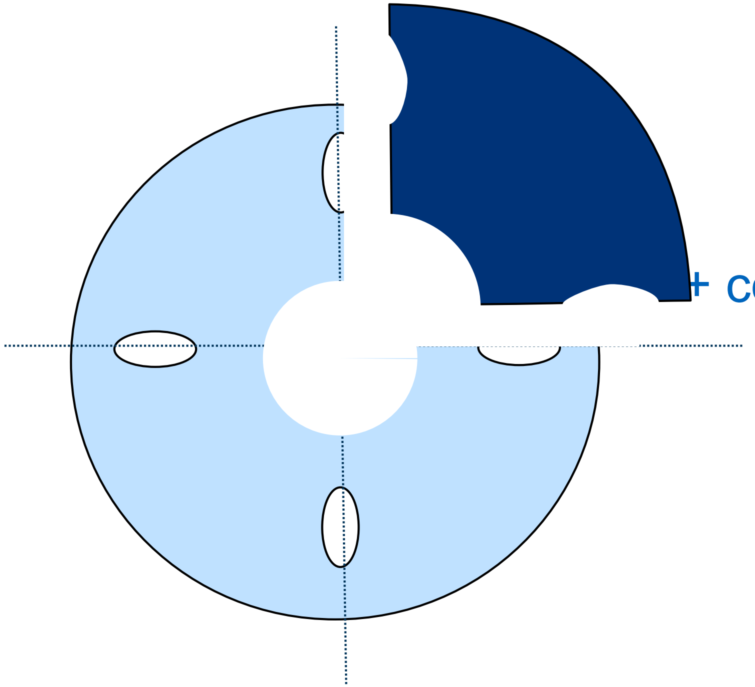


+ compatibility conditions on interface to determine $\boldsymbol{g}_k$

$$\boldsymbol{T}\boldsymbol{u}_L^{(s)} = \boldsymbol{u}_R^{(s+1)} \qquad \boldsymbol{T}\boldsymbol{g}_L^{(s)} = -\boldsymbol{g}_R^{(s+1)}$$

$$\Rightarrow$$

$$\boldsymbol{T}\boldsymbol{u}_{k,L} = \boldsymbol{u}_{k,R}e^{-i\frac{k2\pi}{N}} \qquad \boldsymbol{T}\boldsymbol{g}_{k,L} = -\boldsymbol{g}_{k,R}e^{-i\frac{k2\pi}{N}}$$



$$\boldsymbol{M}\ddot{\boldsymbol{u}}_k + \boldsymbol{K}\boldsymbol{u}_k = \boldsymbol{f}_k + \boldsymbol{g}_k$$

+ compatibility conditions on interface to determine $\boldsymbol{g}_k$

…. and after rearrangement ...

$$\begin{bmatrix} \boldsymbol{M}_{LL} + \boldsymbol{M}_{LR}\boldsymbol{T}e^{i\frac{k2\pi}{N}} + \boldsymbol{T}^T\boldsymbol{M}_{RL}e^{-i\frac{k2\pi}{N}} + \boldsymbol{T}^T\boldsymbol{M}_{RR}\boldsymbol{T} & \boldsymbol{M}_{Li} + \boldsymbol{T}^T\boldsymbol{M}_{Ri}e^{-i\frac{k2\pi}{N}} \\ \boldsymbol{M}_{iL} + \boldsymbol{M}_{iR}\boldsymbol{T}e^{i\frac{k2\pi}{N}} & \boldsymbol{M}_{ii} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}}_{k,L} \\ \ddot{\boldsymbol{u}}_{k,i} \end{bmatrix}$$

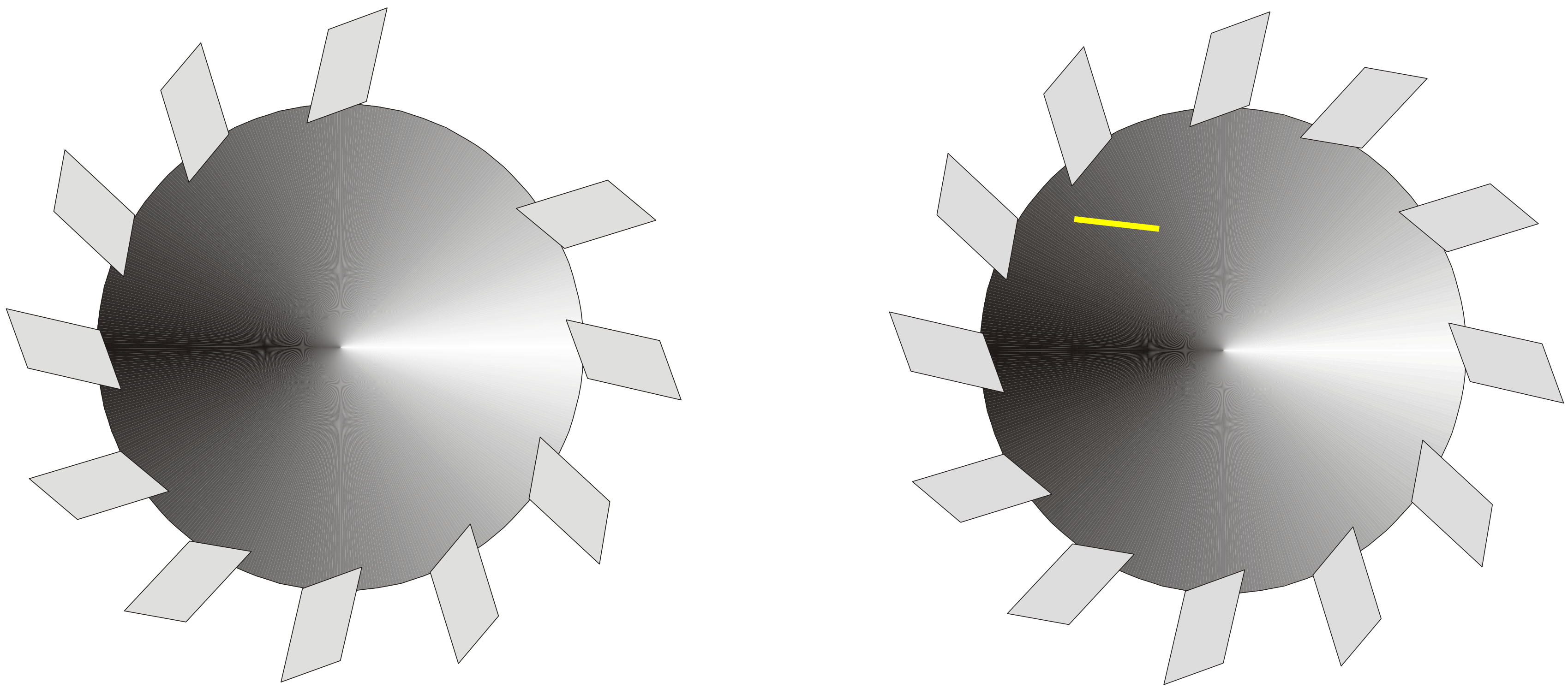
$$+ \begin{bmatrix} \boldsymbol{K}_{LL} + \boldsymbol{K}_{LR}\boldsymbol{T}e^{i\frac{k2\pi}{N}} + \boldsymbol{T}^T\boldsymbol{K}_{RL}e^{-i\frac{k2\pi}{N}} + \boldsymbol{T}^T\boldsymbol{K}_{RR}\boldsymbol{T} & \boldsymbol{K}_{Li} + \boldsymbol{T}^T\boldsymbol{K}_{Ri}e^{-i\frac{k2\pi}{N}} \\ \boldsymbol{K}_{iL} + \boldsymbol{K}_{iR}\boldsymbol{T}e^{i\frac{k2\pi}{N}} & \boldsymbol{K}_{ii} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{k,L} \\ \boldsymbol{u}_{k,i} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{f}_{k,L} + \boldsymbol{T}^T\boldsymbol{f}_{k,R}e^{-i\frac{k2\pi}{N}} \\ \boldsymbol{f}_{k,i} \end{bmatrix}$$

Uncoupled analysis of 1
sector for each harmonic
(commonly available in commercial codes)

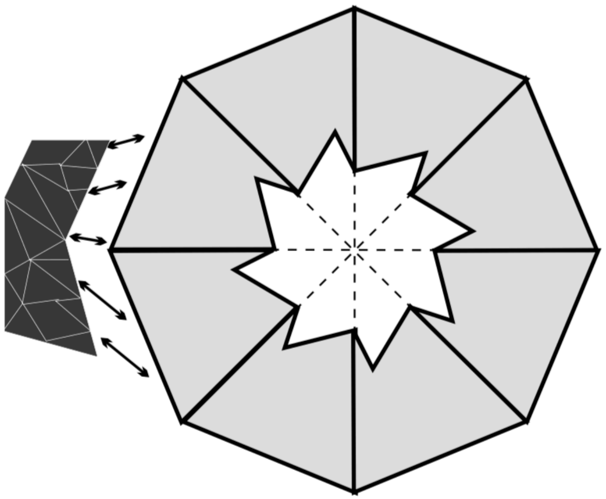
Content

But in practice, the cyclicity is not perfect

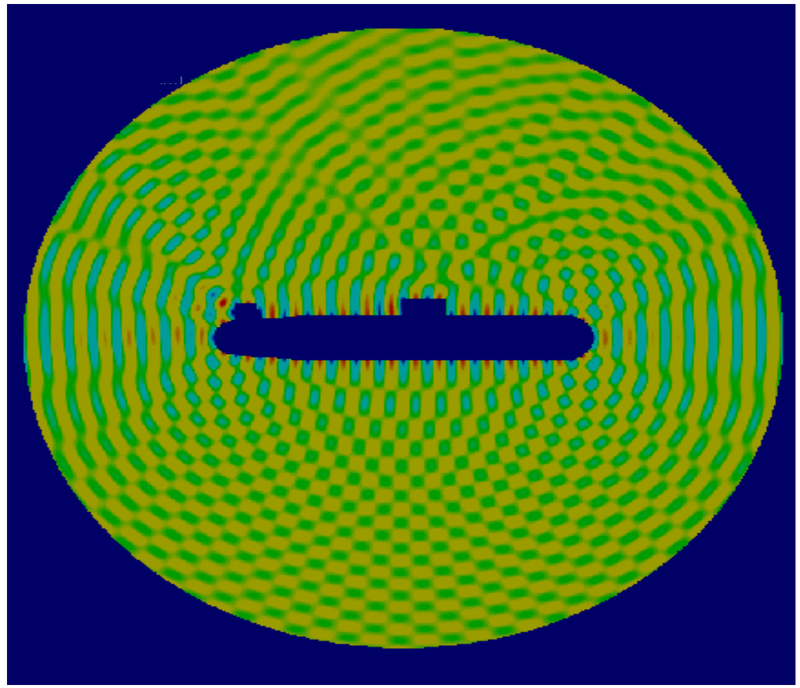


because of defaults, mistuning, dampers or local non-linearities

Quasi-Cyclic systems: 2 domain decomposition approaches



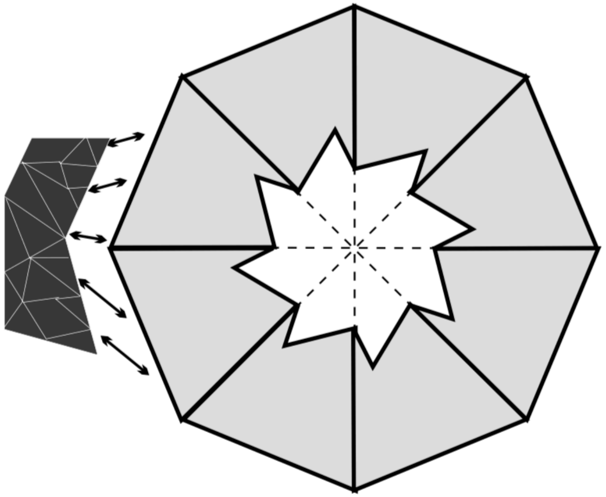
originally proposed for axi-symmetric structures
[Farhat 99, Farhat 02]

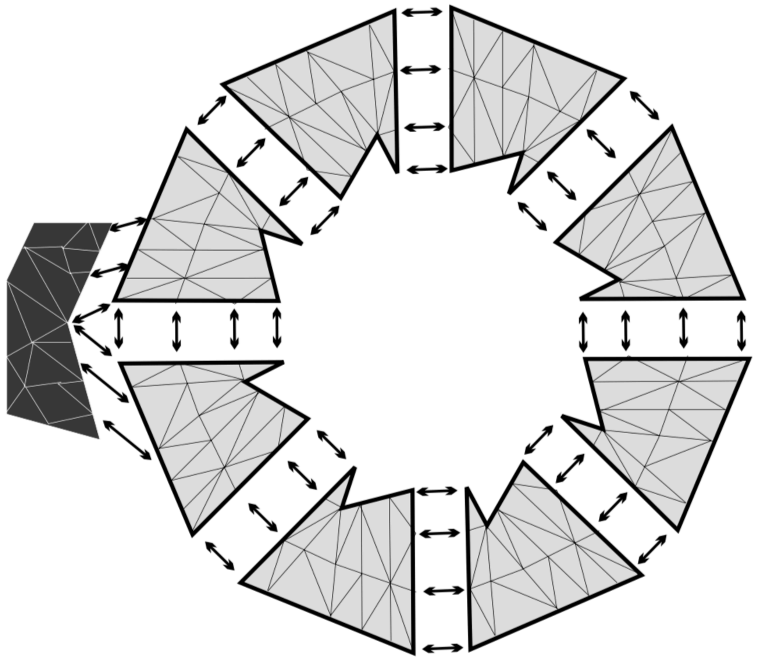


Method 1:

FETI iterations between cyclic part
and non-cyclic part
[Rixen 05]

Quasi-Cyclic systems: 2 domain decomposition approaches





Method 1:

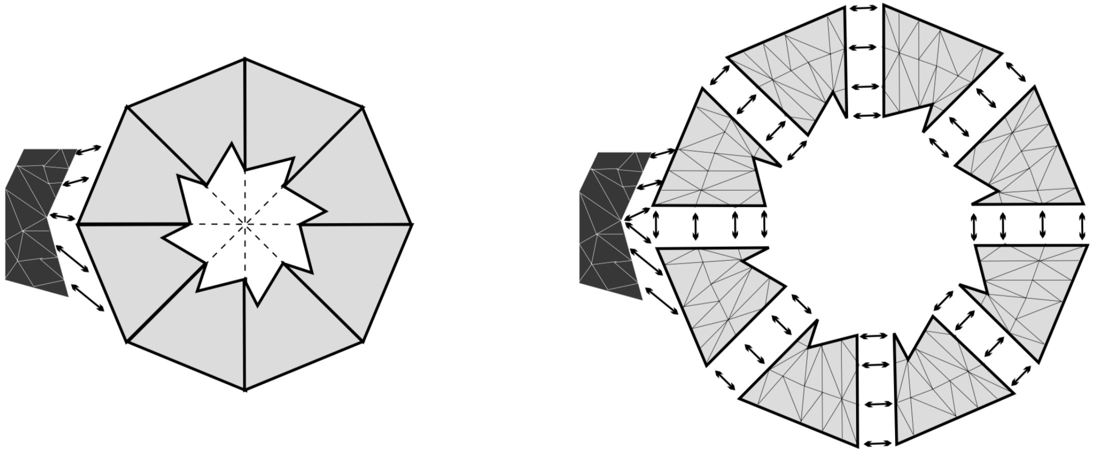
FETI iterations between cyclic part and non-cyclic part [Rixen 05]

Method 2:

FETI applied considering all sectors as separate domains, and use the cyclicity in the preconditioner [Rixen 05]

Research Project WP3 – 1 (ESR 9, Guilherme Jenovencio)

Parallel scalability of the computation of quasi-cyclic turbine models using FETI-type methods



- Evaluate different FETI approaches for quasi-cyclic structures
- Possible extension with more advanced FETI techniques (FETI-DP, FETI-DP)
- Investigate the combination of FETI techniques and multi-harmonic analysis in case of local non-linearity (friction/contact)

Selected References FETI (1)

LeTallec, P. L., De Roeck, Y. H., & Vidrascu, M. (1991). Domain decomposition methods for large linearly elliptic three-dimensional problems. Journal of Computational and Applied Mathematics, 34(1), 93-117.

Farhat, C., & Roux, F. X. (1991). A method of finite element tearing and interconnecting and its parallel solution algorithm. International Journal for Numerical Methods in Engineering, 32(6), 1205-1227.

Rixen, D. J. (1997). *Substructuring and dual methods in structural analysis* (Doctoral dissertation, Publications de la Faculté des Sciences appliquées).

Rixen, D. J. (1998). Dual Schur complement method for semi-definite problems. Contemporary Mathematics, 218(1), 341-348.

Farhat, C., & Mandel, J. (1998). The two-level FETI method for static and dynamic plate problems Part I: An optimal iterative solver for biharmonic systems. Computer methods in applied mechanics and engineering, 155(1), 129-151.

Farhat, C., Chen, P. S., Mandel, J., & Roux, F. X. (1998). The two-level FETI method Part II: Extension to shell problems, parallel implementation and performance results. Computer methods in applied mechanics and engineering, 155(1), 153-179.

Rixen, D. J., & Farhat, C. (1999). A simple and efficient extension of a class of substructure based preconditioners to heterogeneous structural mechanics problems. International Journal for Numerical Methods in Engineering, 44(4), 489-516.

Selected References FETI (2)

Bhardwaj, M., Day, D., Farhat, C., Lesoinne, M., Pierson, K., & Rixen, D. (2000). Application of the FETI method to ASCI problems—scalability results on 1000 processors and discussion of highly heterogeneous problems. International Journal for Numerical Methods in Engineering, 47(1-3), 513-535.

Farhat, C., Lesoinne, M., & Pierson, K. (2000). A scalable dual-primal domain decomposition method. *Numerical linear algebra with applications*, *7*(7-8), 687-714.

Farhat, C., Lesoinne, M., LeTallec, P., Pierson, K., & Rixen, D. (2001). FETI-DP: a dual–primal unified FETI method—part I: A faster alternative to the two-level FETI method. International journal for numerical methods in engineering, 50(7), 1523-1544.

Rixen, D., Le Tallec, P., & Klawonn, A. (2001). A matrix description for the domain decomposition methods of the FETI family. In First MIT Conference on Computational Fluid and Solid Mechanics (pp. 1636-1639). Elsevier: Amsterdam.

Klawonn, A., Widlund, O. B., & Dryja, M. (2002). Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients. SIAM Journal on Numerical Analysis, 40(1), 159-179.

Gosselet, P., Rey, C., & Rixen, D. J. (2003). On the initial estimate of interface forces in FETI methods. Computer methods in applied mechanics and engineering, 192(25), 2749-2764.

Gosselet, P., & Rey, C. (2006). Non-overlapping domain decomposition methods in structural mechanics. Archives of computational methods in engineering, 13(4), 515-572.

Selected References FETI (3)

Klawonn, A., & Widlund, O. B. (2006). Dual-primal FETI methods for linear elasticity. Communications on pure and applied mathematics, 59(11), 1523-1572.

Klawonn, A., & Rheinbach, O. (2007). Robust FETI-DP methods for heterogeneous three dimensional elasticity problems. Computer methods in applied mechanics and engineering, 196(8), 1400-1414.

Lloberas-Valls, O., Rixen, D. J., Simone, A., & Sluys, L. J. (2011). Domain decomposition techniques for the efficient modeling of brittle heterogeneous materials. Computer Methods in Applied Mechanics and Engineering, 200(13), 1577-1590.

Nataf, F., Xiang, H., Dolean, V., & Spillane, N. (2011). A coarse space construction based on local Dirichlet-to-Neumann maps. SIAM Journal on Scientific Computing, 33(4), 1623-1642.

Dolean, V., Nataf, F., Scheichl, R., & Spillane, N. (2012). Analysis of a two-level Schwarz method with coarse spaces based on local Dirichlet-to-Neumann maps. Computational Methods in Applied Mathematics Comput. Methods Appl. Math., 12(4), 391-414.

Lloberas-Valls, O., Rixen, D. J., Simone, A., & Sluys, L. J. (2012). Multiscale domain decomposition analysis of quasi-brittle heterogeneous materials. International Journal for Numerical Methods in Engineering, 89(11), 1337-1366.

Lloberas-Valls, O., Rixen, D. J., Simone, A., & Sluys, L. J. (2012). On micro-to-macro connections in domain decomposition multiscale methods. Computer Methods in Applied Mechanics and Engineering, 225, 177-196.

Selected References FETI (4)

Spillane, N., Dolean, V., Hauret, P., Nataf, F., & Rixen, D. J. (2013). Solving generalized eigenvalue problems on the interfaces to build a robust two-level FETI method. Comptes Rendus Mathematique, 351(5), 197-201.

Spillane, N., & Rixen, D. J. (2013). Automatic spectral coarse spaces for robust finite element tearing and interconnecting and balanced domain decomposition algorithms. International Journal for Numerical Methods in Engineering, 95(11), 953-990.

Spillane, N., Dolean, V., Hauret, P., Nataf, F., Pechstein, C., & Scheichl, R. (2014). Abstract robust coarse spaces for systems of PDEs via generalized eigenproblems in the overlaps. Numerische Mathematik, 126(4), 741-770.

Gosselet, P., Rixen, D., Roux, F. X., & Spillane, N. (2014). Simultaneous-FETI and related block strategies: robust domain decomposition methods for engineering problems. hal.archives-ouvertes.fr

Selected References: Quasi-cyclic

D. L. Thomas. Dynamics of rotationally periodic structures. *International Journal for Numerical Methods in Engineering*, 14(1):81–102, 1979.

D. Rixen and R. Lohman. Efficient computation of eigenmodes of quasi-cyclic structures. In *International Modal Analysis Conference, IMAC-XXIII*, Orlando, FL, February 1-4 2005. SEM.

D. Laxalde, F. Thouverez, and J.-P. Lombard. Dynamical analysis of multi-stage cyclic structures. *Mechanics Research Communications*, 34(4):379–384, 2007.

C. Farhat, U. Hetmaniuk, and D. Rixen. An efficient substructuring method for analyzing structures with major axisymmetric components. In *Structures, Structural Dynamics and Material Conference and Exhibit*, St. Louis, MO, April 12-15 1999. 40st AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference and Exhibit.

C. Farhat and U. Hetmaniuk. A fictitious domain decomposition method for the solution of partially axisymmetric acoustic scattering problems. part i: Dirichlet boundary conditions. *International Journal for Numerical Methods in Engineering* 54(9):1309–1332, 2002.